PHYSICAL REVIEW E **76**, 031706 (2007)

Vibrational Fréedericksz transition in liquid crystals

V. A. Vladimirov*
Department of Mathematics, York University, York, YO10 5DD, United Kingdom

M. Yu. Zhukov†
Department of Mathematics and Mechanics, Rostov State University, 344090, Rostov-on-Don, Russia

(Received 27 December 2006; revised manuscript received 8 August 2007; published 27 September 2007)

Our aim is to show the possibility of a "vibrational Fréedericksz's transition" in nematic liquid crystals (LCs) consisting of asymmetric molecules (in the absence of the reflectional symmetry for the director field). For this purpose we study the effects of the external high-frequency vibrations imposed on LCs by employing the rigorous two-timing asymptotic averaging method to the governing LC equations in their most general form. We restrict our attention only to translational mechanical vibrations of an incompressible LC medium as a whole, so our vibrations are not related to acoustical waves. We show that the averaged "vibrogenic" torque acted on the director field is mathematically equivalent to the torque caused by an external magnetic field. This equivalence leads us to the key conclusion that the high-frequency mechanical vibrations can cause the vibrational Fréedericksz's transition. Our evaluation of the relevant physical parameters shows that this phenomenon can be observed in laboratory experiments.

DOI: 10.1103/PhysRevE.76.031706 PACS number(s): 61.30.Dk

I. INTRODUCTION

Various striking vibrational effects that occur in isotropic fluids are well known; one can count here vibrational convection [1], the motions of solid particles in vibrating fluids [2–5], vibrational transport, and many other interesting phenomena [6,7]. It is also very interesting to consider vibrational effects in liquid crystals (LCs) where one can expect a variety of qualitatively new and specific for the LC media effects. A specific effect we study in this paper is the *vibrational Fréedericksz's transition*, which can occur in a nematic LC medium under the action of high-frequency vibrations.

We consider a LC medium that consists of asymmetric molecules with the centers of gravity not coinciding with their geometrical centers (one can find the description of such liquid crystals including the *Janus liquid crystals* in [8]). The basic mathematical model we use is purely phenomenological; it generalizes the classical Ericksen-Leslie-Parody (ELP) equations [9–15]. To derive it we use the classical approach by Leslie [14,15], Oseen [16], Zocher [17], Frank [18] and Ericksen [19]. Our generalization exploits the absence reflectional invariance ($\boldsymbol{n}\rightarrow-\boldsymbol{n}$) of the director field. It is crucially important that in the ELP equations for the director orientation we also retain the inertial terms. Usually these terms are neglected due to the significant viscosity of LC media. However, we consider high-frequency mechanical vibrations; therefore, those terms are multiplied by the square of this frequency and their impact is always significant (if the frequency is larger than some critical value). For example, it is well accepted that for a proper description of acoustical effects [20,21] or waves of director orientation (see [22–26]) one should keep the inertial terms in the governing equations. In addition, we use some physical "molecular" ideas to evaluate two scalar coefficients in our generalization of the ELP model. Such use is totally legitimate: from the very birth of liquid crystal theory phenomenological models have operated with various auxiliary molecular ideas that have been proven to be the extremely useful heuristic tools. Our final result shows a remarkable degree of indifference of our theory to the heuristic molecular parameters.

Our way to treat vibrations consists of employing a regular two-timing averaged procedure to the generalized equations. As a result we obtain a full system of averaged equations which coincide with the original governing equations, but is complemented by an additional "vibrogenic torque" term in the averaged equations for the director. Here we discover a remarkable mathematical equivalence between two factors: we demonstrate that the averaged equations of a vibrating LC medium (without any magnetic field) after an appropriate change of notations coincide with the equations of (nonvibrating) LCs in the presence of an external magnetic field. This strict mathematical equivalence between the systems of governing equations allows us to predict the existence of the vibrational Fréedericksz's transition and to obtain the threshold frequency by the straightforward replacements in the notations.

In connection with the vibrations of liquid crystals one can also quote a renowned physicist Kapitza ([27], p. 734): "The concept of the vibrational torque can be applied to any body, be it a colloidal particle or a molecule…. Since the nature of the vibrational torque has so far escaped notice of theoretical physics, one has neither experimentally sought for aligned effect on colloidal and molecular particles, which case of a particle's asymmetric shape…." However, as far as we know there are no published research papers devoted to the vibrogenic influence of the high-frequency mechanical vibrations on the molecular orientation.

---

*vv500@york.ac.uk
†zhuk@ns.math.rsu.ru

II. NEMATIC LIQUID CRYSTALS MADE OF ASYMMETRIC MOLECULES

Our aim is to introduce a phenomenological model of a nematic liquid crystal medium that consists of molecules with displaced centers of gravity (not coinciding with their geometrical centres). The main difference from the classical models of nematic LC media is the abolishing of the usually accepted symmetry with respect to the reflections $\boldsymbol{n}\rightarrow-\boldsymbol{n}$ of the director. It is very natural, since the director field $\boldsymbol{n}$ represents the orientation of the LC molecules which are asymmetric in our case. Let us derive the specific internal energy $\hat{w}$ and introduce the correspondent molecular field $\hat{\boldsymbol{h}}$ [9–12]:

$$\hat{w}=w+s\boldsymbol{g}\cdot\boldsymbol{n},\quad \hat{\boldsymbol{h}}=\boldsymbol{\nabla}\frac{\partial\hat{w}}{\partial\boldsymbol{\nabla}\boldsymbol{n}}-\frac{\partial\hat{w}}{\partial\boldsymbol{n}},\tag{1}$$

where s is a new constant and w is the classical version of the specific internal energy. The rigorous derivation takes us back to the use of group theory and to the classical papers by Leslie, Oseen, Zocher, Frank, and Ericksen [14–19]. We reconsider these classical procedure by taking into account the absence of the invariance of w with respect to the reflection $\boldsymbol{n}\rightarrow-\boldsymbol{n}$. Hence, we repeat all required steps retaining the odd in $\boldsymbol{n}$ terms and taking the constant vector of gravity $\boldsymbol{g}$ into account. Additionally we require the linearity with respect to $\boldsymbol{g}$, which comes from the physical idea of the linear dependence of energy on the external gravity field. As a result the scalar term $s\boldsymbol{g}\cdot\boldsymbol{n}$ in $\hat{w}$, Eqs. (1), appears as the only available combination. The coefficient s in this approach is left indefinite and should be found from the experiments or from the underpinning molecular models. In order to define it heuristically one can observe that the expression $s\boldsymbol{g}\cdot\boldsymbol{n}$ represents a potential energy due to the asymmetry of molecules. Simultaneously this term means that there is an *additional specific gravity torque*

$$\boldsymbol{M}_g=\boldsymbol{n}\times(-s\boldsymbol{g})\tag{2}$$

applied to the director field $\boldsymbol{n}$. Using the physical meaning of $\hat{w}$ and $\boldsymbol{M}_g$ one can identify s as the characteristic length related to the asymmetry of molecules. More exactly, we define s as the only available characteristic length: the distance ("shift") between the center of gravity of a molecule and its geometrical center. For the further auxiliary use we also introduce the characteristic size of a molecule μ and the relative displacement s_0:

$$s_0\equiv s/\mu,\quad 1/2\leqslant s_0\leqslant 1/2.$$

The isothermal version of the ELP equations now can be written as (for further use we also consider an externally imposed magnetic field)

$$\begin{aligned}\dot{\boldsymbol{v}}=-\frac{1}{\rho}\boldsymbol{\nabla}p+\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}+\frac{1}{2}\mathrm{rot}[\boldsymbol{n}\times(\gamma_1\boldsymbol{N}+\gamma_2\mathbb{D}\cdot\boldsymbol{n})]\\ -\boldsymbol{h}\times\mathrm{rot}\,\boldsymbol{n}-(\boldsymbol{h}\cdot\boldsymbol{\nabla})\boldsymbol{n},\quad \mathrm{div}\,\boldsymbol{v}=0,\end{aligned}\tag{3}$$

$$\begin{aligned}\boldsymbol{\sigma}=\alpha_4\mathbb{D}+\alpha_1(\boldsymbol{n}\cdot\mathbb{D}\cdot\boldsymbol{n})\boldsymbol{n}\otimes\boldsymbol{n}+\alpha_2\boldsymbol{N}\otimes\boldsymbol{n}+\alpha_3\boldsymbol{n}\otimes\boldsymbol{N}\\ +\alpha_5\mathbb{D}\cdot\boldsymbol{n}\otimes\boldsymbol{n}+\alpha_6\boldsymbol{n}\otimes\mathbb{D}\cdot\boldsymbol{n},\end{aligned}\tag{4}$$

$$\begin{aligned}I\dot{\boldsymbol{\omega}}+\boldsymbol{n}\times(\gamma_1\boldsymbol{N}+\gamma_2\mathbb{D}\cdot\boldsymbol{n}-\boldsymbol{h})=\boldsymbol{M}_g+\boldsymbol{M}_H,\\ \boldsymbol{M}_H=\chi_a(\boldsymbol{n}\times\boldsymbol{H})(\boldsymbol{n}\cdot\boldsymbol{H}),\end{aligned}\tag{5}$$

$$\begin{aligned}\boldsymbol{N}=\dot{\boldsymbol{n}}-\boldsymbol{\Omega}\times\boldsymbol{n},\quad \boldsymbol{\omega}=\boldsymbol{n}\times\dot{\boldsymbol{n}},\quad \boldsymbol{n}^2=1,\quad \boldsymbol{\Omega}=\frac{1}{2}\mathrm{rot}\,\boldsymbol{v},\\ \mathbb{D}=\frac{1}{2}[\boldsymbol{\nabla}\boldsymbol{v}+(\boldsymbol{\nabla}\boldsymbol{v})^*],\end{aligned}\tag{6}$$

$$\begin{aligned}\boldsymbol{h}=\boldsymbol{\nabla}\frac{\partial w}{\partial\boldsymbol{\nabla}\boldsymbol{n}}-\frac{\partial w}{\partial\boldsymbol{n}},\quad \rho w=\rho w_0+\frac{1}{2}K_s(\mathrm{div}\,\boldsymbol{n})^2\\ +\frac{1}{2}K_t(\boldsymbol{n}\cdot\mathrm{rot}\,\boldsymbol{n})^2+\frac{1}{2}K_b(\boldsymbol{n}\times\mathrm{rot}\,\boldsymbol{n})^2.\end{aligned}\tag{7}$$

Here the overdot stays for the material derivative $(\dot{\cdot})=\partial_t+\boldsymbol{v}\cdot\boldsymbol{\nabla}$; $\boldsymbol{v}$, p, $\boldsymbol{g}$, ρ, ω, $\boldsymbol{H}$, and $\mathbb{D}$ are the velocity, pressure, acceleration of gravity, density, the angular velocity of director, magnetic field, and the rate-of-strain tensor, correspondingly; the symbols $\boldsymbol{N}$, $\boldsymbol{h}$, and w denote the "corotational" derivative of the director, the molecular field, and the specific free energy. Notations α_1, α_2, α_3, α_4, α_5, and α_6 correspond to the kinematic viscous coefficients of Leslie; $\gamma_1=(\alpha_3-\alpha_2)$ and $\gamma_2=(\alpha_6-\alpha_5)$ are the coefficients of rotational viscosity; I is the specific moment of inertia; K_s, K_b, and K_t are the elasticity constants (Frank's moduli); χ_a is the magnetic anisotropy. All variables and unknown functions in the equations are considered as dimensionless. However, for the sake of generality, we do not use the dimensionless combinations (the Rayleigh number, Ericksen number, etc.); instead, we introduce the typical (characteristic) values for each physical variable. Notice that for the defining of the molecular specific moment of inertia I in Eqs. (5) one should use the heuristic molecular ideas once again. Here one can refer to [9,20,22–26] where I is defined as

$$I=I_0\mu^2,\tag{8}$$

with the coefficient I_0 depending on the particular geometry of a molecule and the distribution of masses in it; for example, for a rodlike molecule (a homogeneous solid rod of a finite length), $I_0=1/12$. In the equations of liquid crystals I appears as the inertial coefficient in the term $I\dot{\omega}$, Eqs. (5), containing the second time derivative of the director. This term is usually omitted, since it is very small for the physical parameters of the LC displays. However, the high acceleration (due to the high-frequency mechanical vibrations) introduces the square of the frequency as the large multiplier to this term; hence, it can become of a noticeable value. The importance of the retaining of the inertial terms in governing equations of LCs has been already emphasized for acoustical waves [20].

III. VIBRATIONS AND THE TWO-TIMING AVERAGING PROCEDURE

We assume that a LC medium entirely fills a rigid container that performs periodical oscillations in the direction of a fixed unit vector $\boldsymbol{m}$ with the given frequency W and amplitude a. In the equations of motion (3)–(7) such vibrations are mathematically equivalent to the consideration of a fixed (not moving) vessel in an "effective gravity field" $\hat{\boldsymbol{g}}$ containing an oscillating part:

$$\hat{\boldsymbol{g}} \equiv \boldsymbol{g} + aW^2\boldsymbol{m}\cos\tau, \quad \tau \equiv Wt. \tag{9}$$

Here one should notice that the additional oscillating gravity is entirely (mathematically) equivalent to the oscillations of the rigid container. It can be obtained by the transformation of the used exact equations of motion (and we have done it), but we skip this trivial but rather cumbersome procedure since it represents just a manifestation of Einstein's principle of equivalence between the gravity field and acceleration. One can state here even more: if it is not true (if the oscillations of gravity are not equivalent to that of a container), then the used version of governing equations is wrong.

We study the case when the externally enforced frequency W is much higher than the inverse time of the viscous director self-reorientation (exactly the latter scale defines the characteristic time of the Fréedericksz's transition). In this situation the variable τ appears as the *fast time* and we arrive to the classical two-timing setting with the *slow time* t and the *fast time* τ. The first mathematical problem is to derive the averaged (over τ) *slow equations of motion*. The full system of the governing equations that consists of the equations for the director field (5)–(7) and equations for the velocity (3) and (4). The simplest and shortest way to use the averaging procedure is to put the velocity $\boldsymbol{v}(\boldsymbol{x},t)\equiv 0$. For the Fréedericksz's transition it is acceptable since the nature of the effect is statical: it comprises of the turning of the director field only [9–13]. However, we consider the full system of equations in order to show that the presence of vibrational velocity does not alter the averaged equations for the director. We perform the average by employing the mathematically rigorous asymptotic procedure of averaging (of two-scale or two-timing type [5,28–33]) with the large parameter W. Following to the general rules for this procedure [5,31,32] we take the order of magnitude for the amplitude as $a=O(1/W)$. The substitution of $a=U/W$ transforms (9) into

$$\hat{\boldsymbol{g}} = \boldsymbol{g} + UW\boldsymbol{m}\cos\tau, \quad a=\varepsilon U, \quad U=O(1),$$

where we use notations for the fast time τ and the related small parameter ε:

$$\tau = Wt, \quad \varepsilon \equiv \frac{1}{W} \ll 1.$$

The equality $a=\varepsilon U$ should be considered as the definition of the characteristic velocity U. The leading terms in the solutions of Eqs. (3)–(7) are taken as

$$\boldsymbol{v} = \bar{\boldsymbol{v}}(\boldsymbol{x},t) + \tilde{\boldsymbol{v}}(\boldsymbol{x},\tau), \quad \boldsymbol{n} = \bar{\boldsymbol{n}}(\boldsymbol{x},t) + \varepsilon\tilde{\boldsymbol{n}}(\boldsymbol{x},\tau),$$

$$p = \bar{p}(\boldsymbol{x},t) + \frac{1}{\varepsilon}\tilde{p}(\boldsymbol{x},\tau), \quad \bar{F}(\boldsymbol{x},t) \overset{def}{=} \frac{1}{2\pi}\int_0^{2\pi} F(\boldsymbol{x},t,\tau)d\tau, \tag{10}$$

where the overbar over the symbols $\bar{\boldsymbol{v}}$, $\bar{\boldsymbol{n}}$, and $\bar{p}$ stays for the average over τ; $\tilde{\boldsymbol{v}}$, $\tilde{\boldsymbol{n}}$, and $\tilde{p}$ are the purely vibrational terms that represent the 2π-periodic functions of τ having zero averaged values. The last equality in Eqs. (10) gives the definition of the average for any 2π-periodic in τ function $F(\boldsymbol{x},t,\tau)$. All "bar" and "tilde" functions are assumed to be of order 1, so the orders of magnitudes of all unknown functions have been expressed only in terms of ε. In particular, Eqs. (10) take into account that the vibrational velocity is of order 1, which agrees with the accepted order of magnitude aW for the enforced vibrations of velocity field.

Since the director $\boldsymbol{n}$ is a unit vector, the equations [with the error of order $O(\varepsilon^2)$] are valid:

$$\bar{\boldsymbol{n}}\cdot\tilde{\boldsymbol{n}} = 0, \quad \bar{\boldsymbol{n}}\cdot\bar{\boldsymbol{n}} = 1. \tag{11}$$

The equation for the director field (5) we rewrite as

$$I\dot{\boldsymbol{\omega}} + \boldsymbol{n}\times\boldsymbol{Q} + s\boldsymbol{n}\times(\boldsymbol{g} + \varepsilon^{-1}U\boldsymbol{m}\cos\tau) = 0, \tag{12}$$

where

$$\boldsymbol{\omega} \equiv \boldsymbol{n}\times\dot{\boldsymbol{n}}, \quad \boldsymbol{Q} \equiv \gamma_1\boldsymbol{N} + \gamma_2\mathbb{D}\cdot\boldsymbol{n} - \boldsymbol{h} - \chi_a\boldsymbol{H}(\boldsymbol{n}\cdot\boldsymbol{H}).$$

We have introduced the notation $\boldsymbol{Q}$ for the terms that appear in the theory of usual (with the symmetry $\boldsymbol{n}\to-\boldsymbol{n}$) liquid crystals; the additional terms $s\boldsymbol{n}\times(\boldsymbol{g}+\varepsilon^{-1}U\boldsymbol{m}\cos\tau)$ appear due to the asymmetry of molecules and imposed vibrations. The evaluation of the derivatives $\dot{\boldsymbol{n}}$ and $\dot{\boldsymbol{\omega}}$ with the use of Eqs. (10) yields

$$\dot{\boldsymbol{n}} = \bar{\boldsymbol{n}}_t + \tilde{\boldsymbol{n}}_\tau + (\bar{\boldsymbol{v}} + \tilde{\boldsymbol{v}})\cdot\nabla(\bar{\boldsymbol{n}} + \varepsilon\tilde{\boldsymbol{n}}),$$

$$\dot{\boldsymbol{\omega}} = \bar{\boldsymbol{\omega}}_t + \varepsilon^{-1}\tilde{\boldsymbol{\omega}}_\tau + (\bar{\boldsymbol{v}} + \tilde{\boldsymbol{v}})\cdot\nabla(\bar{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}).$$

Taking the τ average of Eq. (12) gives [with the error $O(\varepsilon)$]

$$\tilde{\boldsymbol{\omega}} = \bar{\boldsymbol{n}}\times\tilde{\boldsymbol{n}}_\tau + \bar{\boldsymbol{n}}\times(\tilde{\boldsymbol{v}}\cdot\nabla\bar{\boldsymbol{n}}), \quad \bar{\boldsymbol{\omega}} = \bar{\boldsymbol{n}}\times\mathcal{D}_t\bar{\boldsymbol{n}},$$

$$I\tilde{\boldsymbol{\omega}}_\tau = -sU\bar{\boldsymbol{n}}\times\boldsymbol{m}\cos\tau,$$

$$I\mathcal{D}_t\bar{\boldsymbol{\omega}} + \bar{\boldsymbol{n}}\times\bar{\boldsymbol{Q}} + s\bar{\boldsymbol{n}}\times\boldsymbol{g} = \boldsymbol{M}^{\text{vibr}},$$

where

$$\mathcal{D}_t = \partial_t + \bar{\boldsymbol{v}}\cdot\nabla, \quad \bar{\boldsymbol{N}} = \mathcal{D}_t\bar{\boldsymbol{n}} - \bar{\boldsymbol{\Omega}}\times\bar{\boldsymbol{n}},$$

$$\bar{\boldsymbol{Q}} = \gamma_1\bar{\boldsymbol{N}} + \gamma_2\bar{\mathbb{D}}\cdot\bar{\boldsymbol{n}} - \bar{\boldsymbol{h}} - \chi_a\boldsymbol{H}(\bar{\boldsymbol{n}}\cdot\boldsymbol{H}),$$

$$\boldsymbol{M}^{\text{vibr}} = -sU\overline{\tilde{\boldsymbol{n}}\times\boldsymbol{m}\cos\tau} - I\overline{\tilde{\boldsymbol{v}}\cdot\nabla\tilde{\boldsymbol{\omega}}}.$$

Here $\boldsymbol{M}^{\text{vibr}}$ is the *vibrogenic torque* that appears as the result of the averaged action of vibrations. In order to evaluate it, let us define the vibrational displacement $\tilde{\boldsymbol{\xi}}$:

$$\tilde{\boldsymbol{v}} = \tilde{\boldsymbol{\xi}}_\tau, \quad \bar{\tilde{\boldsymbol{v}}} = 0.$$

Then the vibrational angular velocity $\tilde{\boldsymbol{\omega}}$ of the director is

$$\tilde{\boldsymbol{\omega}} = \bar{\boldsymbol{n}}\times\tilde{\boldsymbol{n}}_\tau + \bar{\boldsymbol{n}}\times(\tilde{\boldsymbol{\xi}}_\tau\cdot\nabla\bar{\boldsymbol{n}}),$$

$$I\tilde{\boldsymbol{\omega}} = -sU\bar{\boldsymbol{n}}\times\boldsymbol{m}\sin\tau. \tag{13}$$

We exclude $\tilde{\boldsymbol{\omega}}$ from these equations and perform the τ averaging, taking into account conditions $\overline{\tilde{\boldsymbol{\omega}}}=0$, $\overline{\tilde{\boldsymbol{n}}}=0$, $\tilde{\boldsymbol{\xi}}=0$:

$$sUI^{-1}\bar{\boldsymbol{n}} \times \boldsymbol{m} \cos \tau = \bar{\boldsymbol{n}} \times \tilde{\boldsymbol{n}} + \bar{\boldsymbol{n}} \times (\tilde{\boldsymbol{\xi}} \cdot \nabla \bar{\boldsymbol{n}}).$$

The vector product of this equation with $\bar{\boldsymbol{n}}$ yields

$$\tilde{\boldsymbol{n}} = -sUI^{-1}[\bar{\boldsymbol{n}}(\bar{\boldsymbol{n}} \cdot \boldsymbol{m}) - \boldsymbol{m}]\cos \tau - \tilde{\boldsymbol{\xi}} \cdot \nabla \bar{\boldsymbol{n}}, \qquad (14)$$

where we have used Eqs. (11) and the equality $\bar{\boldsymbol{n}} \cdot (\tilde{\boldsymbol{\xi}} \cdot \nabla \bar{\boldsymbol{n}}) = \frac{1}{2}(\tilde{\boldsymbol{\xi}} \cdot \nabla \bar{\boldsymbol{n}}^2) = 0$, which also follows from Eqs. (11). Now the substitution of Eq. (14) into Eqs. (13) yields

$$\boldsymbol{M}^{\text{vibr}} = \kappa(\bar{\boldsymbol{n}} \times \boldsymbol{m})(\bar{\boldsymbol{n}} \cdot \boldsymbol{m}), \quad \kappa = \frac{s^2U^2}{2I} = \frac{s_0^2a^2W^2}{2I_0}. \qquad (15)$$

In the calculations we have also used the evident identities

$$\overline{\boldsymbol{m} \times (\tilde{\boldsymbol{\xi}} \cdot \nabla \bar{\boldsymbol{n}})\cos \tau + \tilde{\boldsymbol{\xi}}_\tau \cdot \nabla(\boldsymbol{m} \times \bar{\boldsymbol{n}})\sin \tau}$$
$$= \overline{[\boldsymbol{m} \times (\tilde{\boldsymbol{\xi}} \cdot \nabla \bar{\boldsymbol{n}}) - \tilde{\boldsymbol{\xi}} \cdot \nabla(\boldsymbol{m} \times \bar{\boldsymbol{n}})]\cos \tau},$$

$$\boldsymbol{m} \times (\tilde{\boldsymbol{\xi}} \cdot \nabla \bar{\boldsymbol{n}}) - \tilde{\boldsymbol{\xi}} \cdot \nabla(\boldsymbol{m} \times \bar{\boldsymbol{n}}) = 0.$$

Finally we get the equation for the *slow motion* of the director:

$$I\mathcal{D}_t\bar{\boldsymbol{\omega}} + \bar{\boldsymbol{n}} \times (\gamma_1\bar{\boldsymbol{N}} + \gamma_2\bar{\mathbb{D}} \cdot \bar{\boldsymbol{n}} - \bar{\boldsymbol{h}} + s\boldsymbol{g}) = \bar{\boldsymbol{M}}_H + \boldsymbol{M}^{\text{vibr}},$$

$$\bar{\boldsymbol{\omega}} = \bar{\boldsymbol{n}} \times \mathcal{D}_t\bar{\boldsymbol{n}}, \quad \bar{\boldsymbol{M}}_H = \chi_a(\bar{\boldsymbol{n}} \times \boldsymbol{H})(\bar{\boldsymbol{n}} \cdot \boldsymbol{H}), \qquad (16)$$

with $\boldsymbol{M}^{\text{vibr}}$ defined in Eqs. (15). The averaged equation for the velocity field does coincide with the original equation (3) after the replacements of all functions to their averaged values; we do not present it here for the sake of brevity.

One can see that the averaged equations (16) can be formally obtained from the original governing equations by the changing of all unknown functions to their averaged values and adding to the gravity field a new *vibrogenic* term: $\boldsymbol{g} \to \boldsymbol{g} - (\kappa/s)\boldsymbol{m}(\bar{\boldsymbol{n}} \cdot \boldsymbol{m})$. It is also interesting to notice that the averaged vibrogenic torque (15) *is invariant* with respect to the transformation $\bar{\boldsymbol{n}} \to -\bar{\boldsymbol{n}}$, while the exact torque (2) is *not invariant* with respect to transformation $\boldsymbol{n} \to -\boldsymbol{n}$. In other words, the averaged behavior of asymmetric molecules is symmetric. It can be compared with the similar property of a vibrating pendulum, where the positions "up" and "down" became entirely symmetric after the averaging [5].

IV. MATHEMATICAL EQUIVALENCE BETWEEN MAGNETIC AND VIBROGENIC EFFECTS: VIBRATIONAL FRÉEDERICKSZ's TRANSITION

In the previous section we have shown that the averaged procedure does not change all equations of motion except Eqs. (5) in which the additional averaged vibrogenic torque (15) has appeared. There is a remarkable mathematical equivalence between Eqs. (15) and the original magnetic torque [which has been introduced in Eqs. (5)]:

$$\boldsymbol{M}_H = \chi_a(\boldsymbol{n} \times \boldsymbol{H})(\boldsymbol{n} \cdot \boldsymbol{H}). \qquad (17)$$

One can see that the mutual replacements of notations, according to the rules

$$\boldsymbol{M}_H \leftrightarrow \boldsymbol{M}^{\text{vibr}}, \quad \boldsymbol{H} \leftrightarrow \boldsymbol{H}^{\text{vibr}}, \quad \chi_a \leftrightarrow \chi^{\text{vibr}},$$

$$\boldsymbol{n} \leftrightarrow \bar{\boldsymbol{n}}, \quad \boldsymbol{H}^{\text{vibr}} \equiv \boldsymbol{m}aW, \quad \chi^{\text{vibr}} \equiv \frac{s\mu^2}{2I} = \frac{s_0^2}{2I_0}, \qquad (18)$$

transform either of these expressions (15) or (17) into another. Therefore one can make an important observation that the *whole system of the averaged equations (in the presence of vibrations but without a magnetic field) is mathematically equivalent to the original system (in the presence of a magnetic field but the absence of vibrations).*

This remarkable mathematical equivalence immediately allows us to predict that the *vibrational Fréedericksz's transition* should take place. It gives us even more than that: we can find the threshold frequency for this transition by the mere changes in notations. Indeed, it is known that the critical (threshold) value of the magnetic field for the first Fréedericksz's transition in a liquid crystal layer is

$$|\boldsymbol{H}_c| = \frac{\pi}{d}\sqrt{\frac{K}{\rho\chi_a}}, \qquad (19)$$

where d is the layer thickness, and K is one out of three elasticity constants K_t, K_b, and K_s that correspond to the twist, bend, and splay deformations; the particular constant, which appears in Eq. (19), should be chosen according to the type of Fréedericksz's transition under consideration. The changes in Eq. (19) according to (18) yield the *critical frequency of the vibrational Fréedericksz's transition*:

$$W_c = \frac{\pi}{asd}\sqrt{\frac{2KI}{\rho}} = \frac{\pi}{as_0d}\sqrt{\frac{2KI_0}{\rho}}. \qquad (20)$$

This expression represents the main result of this paper. In order to demonstrate the magnitudes of the involved parameters let us take the following.

(i) $s_0 \approx 10^{-2}$; it is a very small relative shift of the center of gravity (in reality it can be much greater [8]); if one increases it, the critical frequency will be even lower.

(ii) $I_0 \approx 1/12$; we take almost a homogeneous rod-type molecule, the error is negligible until s_0 is small.

(iii) $K \approx 3.8 \times 10^{-12}$ N; the elasticity constants for MBBA are $K_t \approx 3.8 \times 10^{-12}$ N, $K_b \approx 7.5 \times 10^{-12}$ N, $K_s \approx 6.0 \times 10^{-12}$ N [12].

(iv) $d \approx 10^{-4}$ m; it is still greater than the thickness of the LC displays.

(v) $\rho \approx 10^3$ kg/m^3.

For this set of parameters we obtain the critical velocity of vibrations $U_c = aW_c \approx 0.08$ m/s. It means that if the frequency is $f = 120$ Hz ($W = 2\pi f$), then the amplitude is $a \approx 10^{-4}$ m, the density of vibrational energy is $E^{\text{vibr}} = \frac{1}{2}\rho a^2W_c^2 \approx 3$ J/m^3, and the acceleration of vibrations is $aW_c^2 \approx 60$ m/s^2. These estimations show that the *vibrational Fréedericksz's transition can take place for the values of physical parameters that naturally appear in the laboratory experiments.*

V. DISCUSSION

In our model we introduce the additional to ELP term $s\boldsymbol{g} \cdot \boldsymbol{n}$ incorporated into the potential (internal) energy (1).

The first related question is if this term is sufficient to describe fully the action of gravity on the LCs made of asymmetric molecules. We mentioned that the required systematic mathematical study takes us back to the use of group theory and to the classical papers (such as Leslie, Oseen, Zocher, Frank, and Ericksen) [14–19]. However, in our consideration we have identified only the main correction to w, Eqs. (1). We do recognize that the full implementation of this approach can produce more general governing equations where additional constants for both viscosity and elasticity can appear.

The second question is the legitimacy of our auxiliary use of the "molecular" ideas. There are two related issues in the text: the first one is the identification of the parameter s in Eqs. (1) and (2) as the "shift" of the molecular center of gravity; the second issue is the accepted value for molecular specific moment of inertia I, Eq. (8). Both assumptions have a very strong physical background; the second one has been already used by a number of authors [9,20,22–26]. Remarkably, our main result for the threshold frequency (20) does not depend on both μ and s; it demonstrates the strong degree of indifference of our theory to the molecular parameter used. At the same time we think that a systematic statistical treatment of the considered problem represents an important and very challenging research problem. Perhaps, the case of rod-type molecules of a neutral buoyancy that are "swimming" in a surrounded fluid can be easiest to study. In this case the "rotational potential energy" $s\boldsymbol{g}\cdot\boldsymbol{n}$ will naturally appear. As a relevant example one can think about the lyotropic liquid crystals where the structural units are the clusters or micelles [34]. In this connection one can also mention the "living nematic media" [35–37] and the self-swimming microorganisms [38–42]. It is apparent that for both these biological applications a similar vibrational effect can take place.

The next question is purely physical: it is apparent that the torque $\boldsymbol{M}_g=\boldsymbol{n}\times(-s\boldsymbol{g})$, Eq. (2), can cause the rotation of the director field (molecules) and Fréedericksz's transition even in the constant field of gravity. Why in this simple and natural situation has the Fréedericksz's transition not been observed yet? This question can be answered after estimation of the values of the vibrational acceleration and the relevant time scales. One can see that for the set of parameters presented in the text the vibrational acceleration is approximately 6 times greater than gravity. At the same time the structure of the ordinary static torque $\boldsymbol{n}\times(-s\boldsymbol{g})$ differs significantly from the structure of the vibrogenic torque $\boldsymbol{M}^{\text{vibr}}=\kappa(\bar{\boldsymbol{n}}\times\boldsymbol{m})(\bar{\boldsymbol{n}}\cdot\boldsymbol{m})$, so the related rotations of the director field should be very different. It is clear that the presence of the external torque $\boldsymbol{n}\times(-s\boldsymbol{g})$ eventually results in a statical equilibrium with the director parallel to the gravity. However, the related characteristic time of the viscous rotational relaxation T_r is very long, since it is linked to the inverse small parameter s: $T_r=\gamma_1/(s_0\mu|g|)$. For example, for the MBBA the rotational viscosity $\gamma_1\approx10^{-4}$ m^2 s^{-1}. If we take the molecular size $\mu\approx10^{-9}$ m, then $T_r\approx10^6$ s (we have taken $|g|\approx10$ m s^{-2} and $s_0\approx10^{-2}$). So it is very likely that the statical effect can be compensated by other physical factors and the vibrational effects can dominate over the static ones. This question certainly deserves further study.

A key issue to address is the applicability of the two-timing averaging method used. The main danger in the use of the averaging method is the appearance of resonances between the enforced frequency and the self-frequencies of the system [30,33]. However, in the case when the self-frequencies are strongly separated into two (or more) groups, one can consider an intermediate enforced frequency and the averaging method will work well. For example, for the Stephenson-Kapitza pendulum [27] the mechanical self-frequency of the "rigid" pendulum in the gravity field is much smaller than any elastic frequencies of the construction (which do always exist). In the experiments (and in the theory) the pivot's oscillations produce the phenomenon of the "inverted pendulum" when its frequency is much higher than the mechanical one and much lower than those of the elastic nature. The spectrum of self-frequencies for the liquid crystals is similar: there are low frequencies related to the Fréedericksz's transition and much higher acoustical (and some other) self-frequencies. Indeed, the frequency of the rotational viscous relaxation of the director field is (as was already mentioned above) $f_{visc}=K/(2\pi\rho\gamma_1d^2)\approx10^{-2}$ Hz. Exactly this frequency is directly linked to the Fréedericksz's transition that manifests itself in the reorientation of the directors in the entire layer of liquid crystal. It is the reason why in the last formula we take the thickness of the layer d as the characteristic length scale, not the molecular size μ (related to the time scale for the reorientation of a molecule). In contrast, the frequency for the molecular self-oscillations (if we treat each individual molecule as an isolated pendulum in the gravity field) is $f_{pend}=(s_0\mu|g|)/(2\pi I_0\mu^2)\approx17\,000$ Hz (we took $\mu\approx10^{-9}$ m). The frequency of elastic waves is $f_{elast}=[K/(\rho I_0\mu^2)]^{1/2}(2\pi\lambda)^{-1}$, where λ is the wavelength. For the numerical parameters in Sec. V we get $f_{elast}\approx35$ (m s^{-1})λ^{-1} (here λ must be taken in meters), which is still much higher than f_{visc} for all reasonable wavelengths. Therefore in this our paper we consider the notion of "high frequency" as *a frequency which is much higher than the self-frequencies involved in the phenomenon of Fréedericksz's transition and much lower than acoustical, etc., frequencies*. The case when the externally enforced frequency is equal (is in resonance) to some acoustical, etc., very high frequency requires a completely separate theory that is not considered here.

It is also important to mention the effects of boundary conditions. The above theory is developed for the medium vibrating as a whole. It means that in the main motion there are no relative vibrations of the container boundaries with respect to the LC medium. In this particular situation the effect of *steady streaming* [43,44] is trivially absent. These circumstances require a special precaution in the related (and desired) experiments. If in a real experiment (which can be influenced by many additional factors) some significant steady streaming flows also appear, then the observation of Fréedericksz's transition can be more difficult. However, one should take into account that in the very thin LC layers (such as in the LC displays) the viscous effects are so strong that the appearance of any significant steady streaming flows is unlikely.

A very important research direction [45–49] considers the effects of the director reorientations due to the action of the sound and ultrasound in a LC medium. To compare the acoustically driven phenomena with the problem of our paper one should keep in mind that the acoustical phenomena have a much more complicated nature; they are not directly related to the physically simpler and mathematically more rigorous case considered in our paper. More precisely, this relation should be a subject of additional very serious and systematic studies.

Finally, we note that the proposed theory should be viewed only as the first step in the chosen direction. One can consider a more complex model [50] where along with the usual rotation of the director its "spin" has been introduced. Even more sophisticated approaches could be based on the general models of anisotropic media such as Eringen's micropolar model [51] (see also [52], where also the comparison between Eringen's and ELP models are given), the Pleiner-Brand model [53], or the Qian-Sheng model [54] dealing with the order tensor. In these more advanced theories the asymmetry of the basic structural units (molecules) can be introduced naturally by means of the inertia tensor or the order tensor. There are also mesoscopic models of anisotropic media that use hidden variables describing the orientational properties of the medium [55,56]. In particular, [55,56] have shown that the reduction of such general models to the ELP theory is possible only if one accepts the symmetry $\boldsymbol{n} \rightarrow -\boldsymbol{n}$.

ACKNOWLEDGMENTS

We are grateful to Professor G. W. Goodby and Professor E. P. Raynes for their active interest in this research and discussions. This research is partially supported by EPSRC (Research Grants Nos. GR/S96616/01, EP/D055261/1, and EP/D035635/1), Russian Presidential Programme for Distinguished Scientific Schools (Grant No. 5747.2006.01), and INTAS-Airbus project "Vortex Dynamics" (Grant No. 04-80-7297). The authors are grateful to the Hull Institute of Mathematical Sciences and Applications (HIMSA) where this research was initiated and to the Department of Mathematics of the University of York for the providing of excellent conditions for this research.

[1] S. M. Zenkovskaya and I. B. Simonenko, Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza **1**, 51 (1966).
[2] E. J. Hinch and L. C. Nitsche, J. Fluid Mech. **256**, 343 (1993).
[3] O. M. Lavrenteva, Eur. J. Appl. Math. **10**, 251 (1999).
[4] B. A. Lugovtsov and V. L. Sennitskii, Sov. Phys. Dokl. **31**, 530 (1986).
[5] V. A. Vladimirov, J. Math. Fluid Mech. **7**, 397 (2005).
[6] I. I. Blekhman, *Vibrational Mechanics* (World Scientific, Singapore, 2000).
[7] D. V. Lubimov, T. P. Lubimova, and A. A. Cherepanov, *Dynamics of Interfaces in Vibrational Fields* (Fizmatlit, Moscow, 2003) (in Russian).
[8] I. M. Saez and J. W. Goodby, Chem.-Eur. J. **9**, 4869 (2003).
[9] M. J. Stephan and J. P. Straley, Rev. Mod. Phys. **46**, 617 (1974).
[10] S. Chandrasekhar, *Liquid Crystals* (Cambridge University Press, Cambridge, England, 1992).
[11] P. G. de Gennes and J. Prost, *The Physics of Liquid Crystals* (Clarendon Press, Oxford, 1993).
[12] I. Stewart, *The Static and Dynamic Continuum Theory of Liquid Crystals* (Taylor & Francies, 2004).
[13] L. D. Landau and E. M. Lifshitz, *Theory of Elasticity* (Pergamon Press, New York, 1986).
[14] F. M. Leslie, Q. J. Mech. Appl. Math. **19**, 357 (1966).
[15] F. M. Leslie, Arch. Ration. Mech. Anal. **28**, 265 (1968).
[16] C. W. Oseen, Trans. Faraday Soc. **29**, 883 (1933).
[17] H. Zocher, Trans. Faraday Soc. **29**, 945 (1933).
[18] F. C. Frank, Discuss. Faraday Soc. **25**, 19 (1958).
[19] J. L. Ericksen, Arch. Ration. Mech. Anal. **23**, 266 (1966).
[20] A. P. Kapustin and O. A. Kapustina, *Acoustics of Liqud Crystals* (Nauka, Moscow, 1986) (in Russian).
[21] O. A. Kapustina, Mol. Cryst. Liq. Cryst. **112**, 1 (1984).
[22] Zhu Guozhen, Phys. Rev. Lett. **49**, 1332 (1982).
[23] Lin Lei, Shu Changqing, Sheng Juelian, P. M. Lam, and Hung Yun, Phys. Rev. Lett. **49**, 1335 (1982).
[24] P. E. Cladis, H. R. Brand, and P. L. Finn, Phys. Rev. A **28**, 512 (1983).
[25] Xu Gang, Shu Chang-Qing, and Lin Lei, Phys. Rev. A **36**, 277 (1987).
[26] Guo Youjiang and Ou-Yang Zang-can, Phys. Rev. A **40**, 2810 (1989).
[27] P. L. Kapitza, in *Collected Papers of P. L. Kapitza*, edited by D. ter Haar (Pergamon Press, Oxford, 1965), Vol. II (1938–1964).
[28] N. N. Bogoliubov and Y. A. Mitropolskii, *Asymptotic Methods in the Theory of Nonlinear Oscillations* (Gordon and Beach, New York, 1961).
[29] A. H. Nayfeh, *Perturbation Methods* (Wiley, New York, 1973).
[30] J. A. Sanders and F. Verhulst, *Averaging Methods in Nonlinear Dynamical Systems, Vol. 59 of Applied Mathematical Sciences* (Springer, Berlin, 1985).
[31] V. I. Yudovich, Prikl. Mat. Mekh. **62**, 968 (1998).
[32] V. I. Yudovich, Phys. Dokl. **42**, 322 (1997).
[33] V. I. Arnold, V. V. Kozlov, and A. I. Neishtadt, *Mathematical Aspects of Classical and Celestial Mechanics* (Springer, Berlin, 1997).
[34] A. S. Sonin, Usp. Fiz. Nauk **153**, 274 (1987).
[35] R. Kemkemer, D. Kling, D. Kaufmann, and H. Gruler, e-print arXiv:physics/9811049.
[36] H. Gruler, U. Dewald, and M. Eberhardt, Eur. Phys. J. B **11**, 187 (1999).
[37] H. A. Simha and S. Ramaswamyt (unpublished).
[38] J. O. Kessler, J. Fluid Mech. **173**, 191 (1986).
[39] T. J. Pedley, N. A. Hill, and J. O. Kessler, J. Fluid Mech. **195**, 223 (1988).
[40] M. A. Bees, N. A. Hill, and T. J. Pedley, J. Math. Biol. **36**, 269 (1998).
[41] T. J. Pedley and J. O. Kessler, J. Fluid Mech. **212**, 155 (1990).

[42] T. J. Pedley and J. O. Kessler, Annu. Rev. Fluid Mech. **24**, 313 (1992).
[43] J. T. Stuart, J. Fluid Mech. **24**, 673 (1966).
[44] N. Riley, J. Inst. Math. Appl. **3**, 419 (1967).
[45] W. Helfrich, Phys. Rev. Lett. **29**, 1583 (1972).
[46] J.-L. Dion, J. Appl. Phys. **50**, 2965 (1979).
[47] J. V. Selinger, M. S. Spector, V. A. Greanya, B. T. Weslowski, D. K. Shenoy, and R. Shashidhar, Phys. Rev. E **66**, 051708 (2002).
[48] F. Bonetto, E. Anoardo, and R. Kimmich, Chem. Phys. Lett. **361**, 237 (2002).
[49] F. Bonetto and E. Anoardo, Phys. Rev. E **68**, 021703 (2003).
[50] A. I. Leonov and V. S. Volkov, e-print arXiv:cond-mat/0203265.
[51] A. C. Eringen, ARI 50, 73 (1997).
[52] D. Lhuillier and A. D. Rey, J. Non-Newtonian Fluid Mech. **120**, 169 (2004).
[53] E. Jarkova, H. Pleiner, H.-W. Muller, A. Fink, and H. R. Brand, Eur. Phys. J. E **5**, 583 (2001).
[54] T. Qian and P. Sheng, Phys. Rev. E **58**, 7475 (1998).
[55] W. Muschik, C. Papenfuss, and H. Ehrentraut, J. Non-Newtonian Fluid Mech. **96**, 255 (2001).
[56] W. Muschik, C. Papenfuss, and H. Ehrentraut, J. Non-Newtonian Fluid Mech. **119**, 91 (2004).